fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM /;; dk] j?kcj jk[k VdAA

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STUDY PACKAGE

Subject : Mathematics

Topic : Probability

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Probability

There are various phenomena in nature, leading to an outcome, which cannot be predicted apriori e.g. in tossing of a coin, a head or a tail may result. Probability theory aims at measuring the uncertainties of such outcomes.

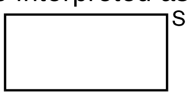
(I) Important terminology:

(i) ***Random Experiment*** **:**

It is a process which results in an outcome which is one of the various possible outcomes that are known to us before hand e.g. throwing of a die is a random experiment as it leads to fall of one of the outcome from {1, 2, 3, 4, 5, 6}. Similarly taking a card from a pack of 52 cards is also a random experiment.

(ii) ***Sample Space :***

It is the set of all possible outcomes of a random experiment e.g. {H, T} is the sample space associated with tossing of a coin.
In set notation it can be interpreted as the universal set.



Solved Example # 1

Write the sample space of the experiment 'A coin is tossed and a die is thrown'.

Solution

The sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

Solved Example # 2

Write the sample space of the experiment 'A coin is tossed, if it shows head a coin tossed again else a die is thrown.

Solution

The sample space S = {HH, HT, T1, T2, T3, T4, T5, T6}

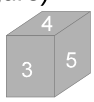
Solved Example # 3

Find the sample space associated with the experiment of rolling a pair of dice (plural of die) once. Also find the number of elements of the sample space.

Sol. Let one die be blue and the other be grey. Suppose '1' appears on blue die and '2' appears on grey die. We denote this outcome by an ordered pair (1, 2). Similarly, if '3' appears on blue die and '5' appears on grey die, we denote this outcome by (3, 5) and so on. Thus, each outcome can be denoted by an ordered pair (x, y), where x is the number appeared on the first die (blue die) and y appeared on the second die (grey die). Thus, the sample space is given by

S = {(x, y) x is the number on blue die and y is the number on grey die}

We now list all the possible outcomes (figure)



	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure

Number of elements (outcomes) of the above sample space is 6 × 6 i.e., 36

Self Practice Problems :

1. A coin is tossed twice, if the second throw results in head, a die is thrown.
Answer {HT, TT, HH1, HH2, HH3, HH4, HH5, HH6, TH1, TH2, TH3, TH4, TH5, TH6}.

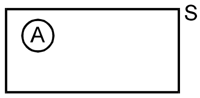
2. An urn contains 3 red balls and 2 blue balls. Write sample space of the experiment 'Selection of a ball from the urn at random'.
Answer {R_1, R_2, R_3, B_1, B_2 }.

Note : -

Here the balls are distinguished from one and other by naming red balls as R_1, R_2 and R_3 and the blue balls as B_1 and B_2.

(iii) ***Event :***

It is subset of sample space. e.g. getting a head in tossing a coin or getting a prime number is throwing a die. In general if a sample space consists 'n' elements, then a maximum of 2^n events can be associated with it.



(iv) ***Complement of event :***

The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\overline{A}$ or A^c.

(v) ***Simple Event :***

If an event covers only one point of sample space, then it is called a simple event e.g. getting a head followed by a tail in throwing of a coin 2 times is a simple event.

(vi) ***Compound Event :***

When two or more than two events occur simultaneously, the event is said to be a compound event. Symbolically $A \cap B$ or AB represent the occurrence of both A & B simultaneously.

Note : "$A \cup B$" or $A + B$ represent the occurrence of either A or B.

Solved Example # 4

Write down all the events of the experiment 'tossing of a coin'.

Solution

S = {H, T}

the events are ϕ , {H}, {T}, {H, T}

Solved Example # 5

A die is thrown. Let A be the event ' an odd number turns up' and B be the event 'a number divisible by 3 turns up'. Write the events (a) A or B (b) A and B

Solution

A = {1, 3, 5}, B = {3, 6}

$\therefore$ A or B = $A \cup B$ = {1, 3, 5, 6}

A and B = $A \cap B$ = {3}

Self Practice Problems :

3. A coin is tossed and a die is thrown. Let A be the event 'H turns up on the coin and odd number turns up on the die' and B be the event ' T turns up on the coin and an even number turns up on the die'. Write the events (a) A or B (b) A and B.

Answer (a) {H1, H3, H5, T2, T4, T6} (b) ϕ

4. In tossing of two coins, let A = {HH, HT} and B = {HT, TT}. Then write the events (a) A or B (b) A and B.

Answer (a) {HH, HT, TT} (b) {HT}

(vii) ***Equally likely Events :***

If events have same chance of occurrence, then they are said to be equally likely.

e.g

(i) In a single toss of a fair coin, the events {H} and {T} are equally likely.

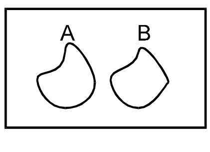
(ii) In a single throw of an unbiased die the events {1}, {2}, {3} and {4}, are equally likely.

(iii) In tossing a biased coin the events {H} and {T} are not equally likely.

(viii) ***Mutually Exclusive / Disjoint / Incompatible Events :***

Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other i.e. both cannot occur simultaneously.

In the vein diagram the events A and B are mutually exclusive. Mathematically, we write $A \cap B = \phi$



Solved Example # 6

In a single toss of a coin find whether the events {H}, {T} are mutually exclusive or not.

Solution

Since $\{H\} \cap \{T\} = \phi$,

$\therefore$ the events are mutually exclusive.

Solved Example # 7

In a single throw of a die, find whether the events {1, 2}, {2, 3} are mutually exclusive or not.

Solution

Since $\{1, 2\} \cap \{2, 3\} = \{2\} \neq \phi$

$\therefore$ the events are not mutually exclusive.

Self Practice Problems :

5. In throwing of a die write whether the events 'Coming up of an odd number' and 'Coming up of an even number' are mutually exclusive or not.
Answer Yes

6. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events :
A : the sum is greater than 8.
B : 2 occurs on either die.
C : the sum is at least 7 and a multiple of 3.
Also, find $A \cap B$, $B \cap C$ and $A \cap C$.
Are (i) A and B mutually exclusive ?
(ii) B and C mutually exclusive ?
(iii) A and C mutually exclusive ?

Ans. A = {(3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)}
B = {(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (2, 1), (2, 3), (2, 4). (2, 5), (2, 6)}
C = {(3, 6), (6, 3), (5, 4), (4, 5), (6, 6)}
$A \cap B = \phi$, $B \cap C = \phi$, $A \cap C$ = {(3, 6), (6, 3), (5, 4), (4, 5), (6, 6)}
(i) Yes (ii) Yes (iii) No.

(ix) *Exhaustive System of Events :*

If each outcome of an experiment is associated with at least one of the events $E_1, E_2, E_3, ...E_n$, then collectively the events are said to be exhaustive. Mathematically we write
$E_1 \cup E_2 \cup E_3.........E_n$ = S. (Sample space)

Solved Example # 8
In throwing of a die, let A be the event 'even number turns up', B be the event 'an odd prime turns up' and C be the event 'a numbers less than 4 turns up'. Find whether the events A, B and C form an exhaustive system or not.

Solution
$A \equiv \{2, 4, 6\}$, $B \equiv \{3, 5\}$ and $C \equiv \{1, 2, 3\}$.
Clearly $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$ = S. Hence the system of events is exhaustive.

Solved Example # 9
Three coins are tossed. Describe
(i) two events A and B which are mutually exclusive
(ii) three events A, B and C which are mutually exclusive and exhaustive.
(iii) two events A and B which are not mutually exclusive.
(iv) two events A and B which are mutually exclusive but not exhaustive.
(v) three events A, B and C which are mutually exclusive but not exhaustive.

Ans. (i) A : "getting at least two heads" B : "getting at least two tails"
(ii) A : "getting at most one heads" B : "getting exactly two heads"
C : "getting exactly three heads"
(iii) A : "getting at most two tails" B : "getting exactly two heads"
(iv) A : "getting exactly one head" B : "getting exactly two heads"
(v) A : " getting exactly one tail" B : "getting exactly two tails"
C : "getting exactly three tails"
[Note : There may be other cases also]

Self Practice Problems :

7. In throwing of a die which of the following pair of events are mutually exclusive ?
(a) the events 'coming up of an odd number' and 'coming up of an even number'
(b) the events 'coming up of an odd number' and 'coming up of a number $\geq$ 4'
Answer (a)

8. In throwing of a die which of the following system of events are exhaustive ?
(a) the events 'an odd number turns up', 'a number $\leq$ 4 turns up' and 'the number 5 turns up'.
(b) the events 'a number $\leq$ 4 turns up', 'a number > 4 turns up'.
(c) the events 'an even number turns up', 'a number divisible by 3 turns up', 'number 1 or 2 turns up' and 'the number 6 turns up'.
Answer (b)

(II) Classical (A priori) Definition of Probability :

If an experiment results in a total of (m + n) outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A', denoted by P(A), is defined by $\frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

i.e. $P(A) = \frac{m}{m+n}$.

We say that odds in favour of 'A' are m : n, while odds against 'A' are n : m.

Note that $P(\overline{A})$ or $P(A')$ or $P(A^c)$, i.e. probability of non-occurrence of A = $\frac{n}{m+n}$ = 1 – P(A)

In the above we shall denote the number of out comes favourable to the event A by n(A) and the total number of out comes in the sample space S by n(S).

$\therefore \quad P(A) = \frac{n(A)}{n(S)}$.

Solved Example # 10

In throwing of a fair die find the probability of the event ' a number ≤ 4 turns up'.

Solution

Sample space S = {1, 2, 3, 4, 5, 6} ; event A = {1, 2, 3, 4}

$\therefore \quad n(A) = 4$ and $n(S) = 6$

$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{4}{6} = \frac{2}{3}$.

Solved Example # 11

In throwing of a fair die, find the probability of turning up of an odd number ≥ 4.

Solution

S = {1, 2, 3, 4, 5, 6}

Let E be the event 'turning up of an odd number ≥ 4'

then E = {5}

$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$.

Solved Example # 12

In throwing a pair of fair dice, find the probability of getting a total of 8.

Solution.

When a pair of dice is thrown the sample space consists

{(1, 1) (1, 2) .... (1, 6)

(2, 1,) (2, 2,)......... (2, 6)

....

....

(6, 1), (6, 2) .... (6, 6)}

Note that (1, 2) and (2, 1) are considered as separate points to make each outcome as equally likely.

To get a total of '8', favourable outcomes are, (2, 6) (3, 5) (4, 4) (5, 3) and (6, 2).

Hence required probability = $\frac{5}{36}$

Solved Example # 13

A four digit number is formed using the digits 0, 1, 2, 3, 4 without repetition. Find the probability that it is divisible by 4

Solution

Total 4 digit numbers formed $\boxed{}\boxed{}\boxed{}\boxed{}$

4 × 4 × 3 × 2 = 96

Each of these 96 numbers are equally likely & mutually exclusive of each other.

Now, A number is divisible by 4, if last two digits of the number is divisible by 4

Hence we can have

		0	4	→	first two places can be filled in 3 × 2 = 6 ways
		1	2	→	first two places can be filled in 2 × 2 = 4 ways
		2	0	→	6 ways
		2	4	→	4 ways
		3	2	→	4 ways
		4	0	→	6 ways

Total number of ways 30 ways

probability = $\frac{\text{favorable outcomes}}{\text{Total outcomes}} = \frac{30}{96} = \frac{5}{16}$ Ans.

Self Practice Problems :

9. A bag contains 4 white, 3 red and 2 blue balls. A ball is drawn at random. Find the probability of the event (a) the ball drawn is white or red (b) the ball drawn is white as well as red.

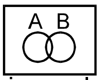
Answer (a) 7/9 (b) 0

10. In throwing a pair of fair dice find the probability of the events ' a total of of less than or equal to 9"'.
Answer 5/36.

(III) Addition theorem of probability :

If 'A' and 'B' are any two events associated with an experiment, then

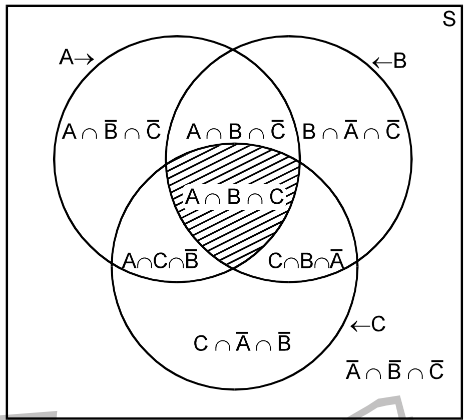
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



De Morgan's Laws : If A & B are two subsets of a universal set U, then
(a) $(A \cup B)^c = A^c \cap B^c$
(b) $(A \cap B)^c = A^c \cup B^c$

Distributive Laws :
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For any three events A, B and C we have the figure



(i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
(ii) P (at least two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$
(iii) P(exactly two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$
(iv) P(exactly one of A, B, C occur) =
$P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Note : If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive, i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$.
However the converse of this is not true.

Solved Example # 14
A bag contains 4 white, 3red and 4 green balls. A ball is drawn at random. Find the probability of the event 'the ball drawn is white or green'.

Solution
Let A be the event 'the ball drawn is white' and B be the event 'the ball drawn is green'.

P(The ball drawn is white or green) $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{8}{11}$

Solved Example # 15
In throwing of a die, let A be the event 'an odd number turns up', B be the event 'a number divisible by 3 turns up' and C be the event 'a number $\leq$ 4 turns up'. Then find the probability that exactly two of A, B and C occur.

Solution
Event $A = \{1, 3, 5\}$, event $B = \{3, 6\}$ and event $C = \{1, 2, 3, 4\}$
$\therefore$ $A \cap B = \{3\}$, $B \cap C = \{3\}$, $A \cap C = \{1, 3\}$ and $A \cap B \cap C = \{3\}$.
Thus P(exactly two of A, B and C occur)
$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$

$= \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{2}{6} - 3 \times \dfrac{1}{6} = \dfrac{1}{6}$

Self Practice Problems :

11. In throwing of a die, let A be the event 'an odd number turns up', B be the event 'a number divisible by 3 turns up' and C be the event 'a number $\leq$ 4 turns up'. Then find the probability that atleast two of A, B and C occur. **Answer** $\dfrac{1}{3}$

12. In the problem number 11, find the probability that exactly one of A, B and C occurs. **Answer** $\dfrac{2}{3}$

(IV) Conditional Probability

If A and B are two events, then $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$.
Note that for mutually exclusive events $P(A/B) = 0$.

Solved Example # 16

If $P(A/B) = 0.2$ and $P(B) = 0.5$ and $P(A) = 0.2$. Find $P(A \cap \overline{B})$.

Solution.

$P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow$ $P(A \cap B) = 0.1$

From given data,

$P(A \cap \overline{B}) = 0.1$

Solved Example # 17

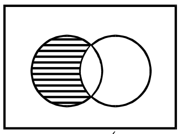
If $P(A) = 0.25$, $P(B) = 0.5$ and $P(A \cap B) = 0.14$, find probability that neither 'A' nor 'B' occurs. Also find $P\left(A \cap \overline{B}\right)$

Solution

We have to find $P\left(\overline{A} \cap \overline{B}\right) = 1 - P(A \cup B)$ (by De-Morgan's law)

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

putting data we get, $P\left(\overline{A} \cap \overline{B}\right) = 0.39$



The shaded region denotes the simultaneous occurrence of A and $\overline{B}$

Hence $P\left(A \cap \overline{B}\right) = P(A) - P(A \cap B) = 0.11$

Self Practice Problem:-

13. If $P(\overline{A} / \overline{B}) = 0.2$, $P(A \cup B) = 0.9$, then find $P(A \cap \overline{B})$?

Ans. 0.4

5. Independent and dependent events

If two events are such that occurence or non-occurence of one does not affect the chances of occurence or non-occurence of the other event, then the events are said to be independent. Mathematically : if $P(A \cap B) = P(A)\,P(B)$, then A and B are independent.

Note: (i) If A and B are independent, then (a) A' and B' are independent, (b) A and B' are independent and (c) A' and B are independent.

(ii) If A and B are independent, then $P(A / B) = P(A)$.

If events are not independent then they are said to be dependent.

Independency of three or more events

Three events A, B & C are independent if & only if all the following conditions hold :

$P(A \cap B) = P(A) . P(B)$; $P(B \cap C) = P(B) . P(C)$

$P(C \cap A) = P(C) . P(A)$; $P(A \cap B \cap C) = P(A) . P(B) . P(C)$

i.e. they must be independent in pairs as well as mutually independent.

Similarly for n events $A_1, A_2, A_3, \ldots\ldots A_n$ to be independent, the number of these conditions is equal to ${}^nC_2 + {}^nC_3 + \ldots\ldots + {}^cC_n = 2^n - n - 1$.

Solved Example # 18

In drawing two balls from a box containing 6 red and 4 white balls without replacement, which of the following pairs is independent ?

(a) Red on first draw and red on second draw

(b) Red on first draw and white on second draw

Solution

Let E be the event 'Red on first draw', F be the event 'Red on second draw' and G be the event 'white on second draw'.

$P(E) = \dfrac{6}{10}$, $P(F) = \dfrac{6}{10}$, $P(G) = \dfrac{4}{10}$

(a) $P(E \cap F) = \dfrac{{}^6P_2}{{}^{10}P_2} = \dfrac{1}{3}$

$P(E) . P(F) = \dfrac{3}{5} \times \dfrac{3}{5} = \dfrac{9}{25} \neq \dfrac{1}{3}$

$\therefore$ E and F are not independent

(b) $P(E) . P(G) = \dfrac{6}{10} \times \dfrac{4}{10} = \dfrac{6}{25}$

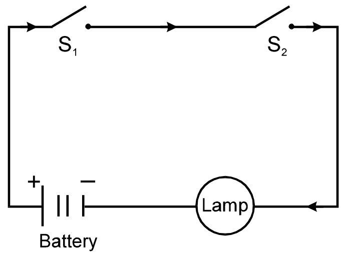
$$P(E \cap G) = \frac{{}^6P_1 \times {}^4P_1}{{}^{10}P_2} = \frac{4}{15}$$

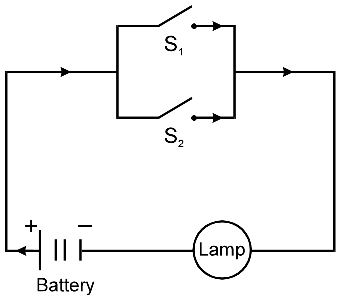
$\therefore$ P(E) . P(G) $\neq$ P(E $\cap$ G)

$\therefore$ E and G are not independent

Solved Example # 19

If two switches S_1 and S_2 have respectively 90% and 80% chances of working. Find the probabilities that each of the following circuits will work.





Solution

Consider the following events :

A = Switch S_1 works,

B = Switch S_2 works,

We have,

$$P(A) = \frac{90}{100} = \frac{9}{10} \text{ and } P(B) = \frac{80}{100} = \frac{8}{10}$$

(i) The circuit will work if the current flows in the circuit. This is possible only when both the switches work together. Therefore,

Required probability

$= P(A \cap B) = P(A)\,P(B)$ [$\because$ A and B are independent events]

$$= \frac{9}{10} \times \frac{8}{10} = \frac{72}{100} = \frac{18}{25}$$

(ii) The circuit will work if the current flows in the circuit. This is possible only when at least one of the two switches S_1, S_2 works. Therefore,

Required Probability

$= P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$ [$\because$ A, Bare independent events]

$$= 1 - \left(1 - \frac{9}{10}\right)\left(1 - \frac{8}{10}\right)$$

$$= 1 - \frac{1}{10} \times \frac{2}{10} = \frac{49}{50}$$

Solved Example # 20

A speaks truth in 60% of the cases and b in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

Solution

Let E be the event that A speaks truth and F be the event that B speaks truth. Then E and F are independent events such that

$$P(E) = \frac{60}{100} = \frac{3}{5} \text{ and } P(F) = \frac{90}{100} = \frac{9}{10}$$

A and B will contradict each other in narrating the same fact in the following mutually exclusive ways :

(i) A speaks truth and B tells a lie i.e. $E \cap \overline{F}$

(ii) A tells a lie and B speaks truth lie i.e. $\overline{E} \cap F$

$\therefore$ P(A and B contradict each other)

$= P(\text{I or II}) = (\text{I} \cup \text{II})$

$= P[(E \cap \overline{F}) \cup (\overline{E} \cap F)]$

$= P(E \cap \overline{F}) + P(\overline{E} \cap F)$ [$\because$ $E \cap \overline{F}$ and $\overline{E} \cap F$ are mutually exclusive]

$= P(E)\,P(\overline{F}) + P(\overline{E})\,P(F)$ [$\because$ E and F are in dep.]

$$= \frac{3}{5} \times \left(1 - \frac{9}{10}\right) + \left(1 - \frac{3}{5}\right) \times \frac{9}{10} = \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{21}{50}$$

Solved Example # 21

An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting

(i) 2 red balls (ii) 2 blue balls (iii) one red and one blue ball

Ans. (i) $\frac{49}{121}$ (ii) $\frac{16}{121}$ (iii) $\frac{56}{121}$

Solved Example # 22

Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
(i) the problem is solved (ii) exactly one of them solves the problem.

Ans. (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$

Solved Example # 23
A box contains 5 bulbs of which two are defective. Test is carried on bulbs one by one untill the two defective bulbs are found out. Find the probability that the process stops after
(i) Second test (ii) Third test

Solution
(i) Process will stop after second test. Only if the first and second bulb are both found to be defective

probability = $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$ (Obviously the bulbs drawn are not kept back.)

(ii) Process will stop after third test when either

(a) DND $\rightarrow$ $\frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{10}$ Here 'D' stands for defective

or (b) NDD $\rightarrow$ $\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$ and 'N' is for not defective.

or (c) NNN $\rightarrow$ $\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

hence required probability = $\frac{3}{10}$

Solved Example # 24

If E_1 and E_2 are two events such that $P(E_1) = \frac{1}{4}$; $P(E_2) = \frac{1}{2}$; $P\left(\frac{E_1}{E_2}\right) = \frac{1}{4}$, then choose the correct options.
(i) E_1 and E_2 are independent (ii) E_1 and E_2 are exhaustive
(iii) E_1 and E_2 are mutually exclusive (iv) E_1 & E_2 are dependent

Also find $P\left(\frac{\overline{E_1}}{E_2}\right)$ and $\left(\frac{E_2}{\overline{E_1}}\right)$

Solution

Since $\left(\frac{E_2}{E_1}\right) = P(E_1)$ $\Rightarrow$ E_1 and E_2 are independent of each other
Also since $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1) \,.\, P(E_2) \neq 1$
Hence events are not exhaustive. Independent events can't be mutually exclusive.
Hence only (i) is correct
Further since E_1 & E_2 are independent; E1 and $\overline{E_2}$ or $\overline{E_1}$, E_2 are $\overline{E_1}$, $\overline{E_2}$ are also independent.

Hence $P\left(\frac{\overline{E_1}}{E_2}\right) = P\left(\overline{E_1}\right) = \frac{3}{4}$ and $P\left(\frac{E_2}{\overline{E_1}}\right) = P\,(E_2) = \frac{1}{2}$

Solved Example # 25
If cards are drawn one by one from a well shuffled pack of 52 cards without replacement, until an ace appears, find the probability that the fourth card is the first ace to appear.

Solution

Probability of selecting 3 non-Ace and 1 Ace out of 52 cards is equal to $\frac{{}^{48}C_3 \times {}^4C_1}{{}^{52}C_4}$

Since we want 4th card to be first ace, we will also have to consider the arrangement, Now 4 cards in sample space can be arranged in 4! ways and, favorable they can be arranged in 3 ! ways as we want 4th position to be occupied by ace

Hence required probability = $\frac{{}^{48}C_3 \times {}^4C_1}{{}^{52}C_4} \times \frac{3\,!}{4\,!}$

Aliter : 'NNNA' is the arrangement than we desire in taking out cards, one by one

Hence required chance is $\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{4}{49}$

Self Practice Problems :

14. In throwing a pair of dies find the probability of getting an odd number on the first die and a total of 7 on both the dies.

Answer $\frac{1}{12}$

15. In throwing of a pair of dies, find the probability of getting a boublet or a total of 4.

Answer $\frac{2}{9}$

16. A bag contains 8 marbles of which 3 are blue and 5 are red. One marble is drawn at random, its colour is noted and the marble is replaced in the bag. A marble is again drawn from the bag and its colour is noted. Find the probability that the marbles will be
(i) blue followed by red (ii) blue and red in any order
(iii) of the same colour.

Ans. (i) $\frac{15}{64}$ (ii) $\frac{15}{32}$ (iii) $\frac{17}{32}$

17. A coin is tossed thrice. In which of the following cases are the events E and F independent ?
(i) E : "the first throw results in head".
F : "the last throw result in tail".
(ii) E : "the number of heads is two".
F : "the last throw result in head".
(iii) E : "the number of heads is odd ".
F : "the number of tails is odd".
Ans. (i)

6. Binomial Probability Theorem

If an experiment is such that the probability of success or failure does not change with trials, then the probability of getting exactly r success in n trials of an experiment is ${}^nC_r\, p^r\, q^{n-r}$, where 'p' is the probability of a success and q is the probability of a failure. Note that p + q = 1.

Solved Example 26
A pair of dice is thrown 5 times. Find the probability of getting a doublet twice.
Solution
In a single throw of a pair of dice probability of getting a doublet is $\frac{1}{6}$

con sidering it to be a success, $p = \frac{1}{6}$

$\therefore \quad q = 1 - \frac{1}{6} = \frac{5}{6}$
number of success r = 2

$\therefore \quad P(r = 2) = {}^5C_2\, p^2\, q^3 = 10 \,.\, \left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$

Solved Example # 27
A pair of dice is thrown 4 times. If getting 'a total of 9' in a single throw is considered as a success then find the probability of getting 'a total of 9' thrice.
Solution
p = probability of getting 'a total of 9' = $\frac{4}{36} = \frac{1}{9}$

$\therefore \quad q = 1 - \frac{1}{9} = \frac{8}{9}$
r = 3, n = 4

$\therefore \quad P(r = 3) = {}^4C_3\, p^3\, q = 4 \times \left(\frac{1}{9}\right)^3 . \frac{8}{9} = \frac{32}{6561}$

Solved Example # 28
In an examination of 10 multiple choice questions (1 or more can be correct out of 4 options). A student decides to mark the answers at random. Find the probability that he gets exactly two questions correct.
Solution
A student can mark 15 different answers to a MCQ with 4 option i.e. ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

Hence if he marks the answer at random, chance that his answer is correct = $\frac{1}{15}$ and being incorrecting

$\frac{14}{15}$. Thus $\quad p = \frac{1}{15}$, $q = \frac{14}{15}$.

$$P \text{ (2 success)} = {}^{10}C_2 \times \left(\frac{1}{15}\right)^2 \times \left(\frac{14}{15}\right)^8$$

Solved Example # 29

A family has three children. Event 'A' is that family has at most one boy, Event 'B' is that family has at least one boy and one girl, Event 'C' is that the family has at most one girl. Find whether events 'A' and 'B' are independent. Also find whether A, B, C are independent or not.

Solution

A family of three children can have
(i) All 3 boys (ii) 2 boys + 1 girl (iii) 1 boy + 2 girls (iv) 3 girls

(i) $P \text{ (3 boys)} = {}^3C_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ (Since each child is equally likely to be a boy or a girl)

(ii) $P \text{ (2 boys +1girl)} = {}^3C_1 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{3}{8}$ (Note that there are three cases BBG, BGB, GBB)

(iii) $P \text{ (1 boy + 2 girls)} = {}^3C_2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = \frac{3}{8}$

(iv) $P \text{ (3 girls)} = \frac{1}{8}$

Event 'A' is associated with (iii) & (iv). Hence $P(A) = \frac{1}{2}$

Event 'B' is associated with (ii) & (iii). Hence $P(B) = \frac{3}{4}$

Event 'C' is associated with (i) & (ii). Hence $P(C) = \frac{1}{2}$

$P(A \cap B) = P(iii) = \frac{3}{8} = P(A) \,.\, P(B)$. Hence A and B are independent of each other

$P(A \cap C) = 0 \neq P(A) \,.\, P(C)$. Hence A, B, C are not independent

Self Practice Problems :

18. A box contains 2 red and 3 blue balls. Two balls are drawn successively without replacement. If getting 'a red ball on first draw and a blue ball on second draw' is considered a success, then find the probability of 2 successes in 3 performances.

Answer .189

19. Probability that a bulb produced by a factory will fuse after an year of use is 0.2. Find the probability that out of 5 such bulbs not more than 1 bulb will fuse after an year of use.

Answer $\frac{2304}{3125}$

7. Expectation :

If a value M_i is associated with a probability of p_i , then the expectation is given by $\sum p_iM_i$.

Solved Example # 30

There are 100 tickets in a raffle (Lottery). There is 1 prize each of Rs. 1000/-, Rs. 500/- and Rs. 200/-. Remaining tickets are blank. Find the expected price of one such ticket.

Solution

Expectation = $\sum p_iM_i$ outcome of a ticket can be

	p_i	M_i	p_iM_i
(i) I prize	$\frac{1}{100}$	1000	10
(ii) II prize	$\frac{1}{100}$	500	5
(iii) III prize	$\frac{1}{100}$	200	2
(iv) Blank	$\frac{97}{100}$	0	0

$\sum p_iM_i = 17$

Hence expected price of one such ticket Rs. 17

Solved Example # 31

A purse contains four coins each of which is either a rupee or two rupees coin. Find the expected value of a coin in that purse.

Solution

Various possibilities of coins in the purse can be

	p_i	M_i	p_iM_i
(i) 4 1 rupee coins	$\frac{1}{16}$	4	$\frac{4}{16}$
(ii) 3 one Rs. + 1 two Rs.	$\frac{4}{16}$	5	$\frac{20}{16}$
(iii) 2 one Rs. + 2 two Rs.	$\frac{6}{16}$	6	$\frac{36}{16}$
(iv) 1 one Rs. + 3 two Rs.	$\frac{4}{16}$	7	$\frac{28}{16}$
(iv) 4 two Rs.	$\frac{1}{16}$	8	$\frac{8}{16}$
			6 / -

Note that since each coin is equally likely to be one Rs. or two Rs. coin, the probability is determined using Binomial probability; unlike the case when the purse contained the coins with all possibility being equally likely, where we take $p_i = \frac{1}{5}$ for each.

Hence expected value is Rs. 6/-

Self Practice Problems :

20. From a bag containing 2 one rupee and 3 two rupee coins a person is allowed to draw 2 coins indiscriminately; find the value of his expectation.

Ans. Rs. 3.20

8. Total Probability Theorem

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$ and the probabilities $P(A/B_1), P(A/B_2) \ldots P(A/B_n)$ are known, then

$$P(A) = \sum_{i=1}^{n} P(B_i) \,.\, P(A/B_i)$$

Solved Example # 32

Box - I contains 5 red and 4 white balls whilst box - II contains 4 red and 2 white balls. A fair die is thrown. If it turns up a multiple of 3, a ball is drawn from box - I else a ball is drawn from box - II. Find the probability that the ball drawn is white.

Solution

Let A be the event 'a multiple of 3 turns up on the die' and R be the event 'the ball drawn is white' then P (ball drawn is white)

$$= P(A) \,.\, P(R / A) + P(\overline{A})\; P(R / \overline{A})$$

$$= \frac{2}{6} \times \frac{4}{9} + \left(1 - \frac{2}{6}\right) \frac{2}{6} = \frac{10}{27}$$

Solved Example # 33

Cards of an ordinary deck of playing cards are placed into two heaps. Heap - I consists of all the red cards and heap - II consists of all the black cards. A heap is chosen at random and a card is drawn, find the probability that the card drawn is a king.

Solution

Let I and II be the events that heap - I and heap - II are choosen respectively. Then

$$P(I) = P(II) = \frac{1}{2}$$

Let K be the event 'the card drawn is a king'

$\therefore \quad P(K / I) = \frac{2}{26} \quad$ and $\quad P(K / II) = \frac{2}{26}$

$\therefore \quad P(K) = P(I)\, P(K / I) + P(II)\, P(K / II) = \frac{1}{2} \times \frac{2}{26} + \frac{1}{2} \times \frac{2}{26} = \frac{1}{13}$.

Self Practice Problems :

21. Box - I contains 3 red and 2 blue balls whilest box - II contains 2 red and 3 blue balls. A fair coin is tossed. If it turns up head, a ball is drawn from box - I, else a ball is drawn from box - II . Find the probability that the ball drawn is red.

Answer $\frac{1}{2}$

22. There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. Find the probability of selecting a brilliant student.

Answer $\frac{17}{125}$.

9. Bayes' Theorem :

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$ and the probabilities $P(A/B_1), P(A/B_2) \ldots P(A/B_n)$ are known, then

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

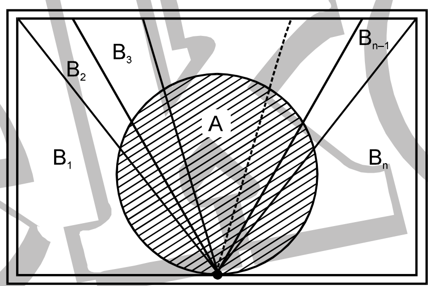
Proof :
The event A occurs with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, \ldots, B_n$

$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots \cup (A \cap B_n)$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A \cap B_i)$$

Note: $A \equiv$ event what we have ; B_i = event what we want ;

Now,
$P(A \cap B_i) = P(A) \cdot P(B_i/A) = P(B_i) \cdot P(A/B_i)$



$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{P(A)} = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(A \cap B_i)}$$

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum P(B_i) \cdot P(A/B_i)}$$

Solved Example # 34

Pal's gardener is not dependable, the probability that he will forget to water the rose bush is $\frac{2}{3}$. The rose bush is in questionable condition any how, if watered the probability of its withering is $\frac{1}{2}$, if not watered, the probability of its withering is $\frac{3}{4}$. Pal went out of station and upon returning, he finds that the rose bush has withered, what is the probability that the gardener did not water the bush.

[Here result is known that the rose bush has withered, therefore. Bayes's theorem should be used]

Solution

Let A = the event that the rose bush has withered
Let A_1 = the event that the gardener did not water.
A_2 = the event that the gardener watered.
By Bayes's theorem required probability,

$$P(A_1/A) = \frac{P(A_1) \cdot P(A/A_1)}{P(A_1) \cdot P(A/A_1) + P(A_2) \cdot P(A/A_2)} \quad \ldots\ldots(i)$$

Given, $P(A_1) = \frac{2}{3}$ $\quad \therefore \quad$ $P(A_2) = \frac{1}{3}$

$P(A/A_1) = \frac{3}{4}$, $P(A/A_2) = \frac{1}{2}$

From (1), $P(A_1/A) = \dfrac{\frac{2}{3}\cdot\frac{3}{4}}{\frac{2}{3}\cdot\frac{3}{4}+\frac{1}{3}\cdot\frac{1}{2}} = \dfrac{6}{6+2} = \dfrac{3}{4}$

Solved Example # 35

There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. A student is a chosen and is found to be brilliant, find the probability that the chosen student is from class XI.

Solution

Let E and F be the events 'Class XI is chosen' and 'Class XII is chosen' respectively.

Then $P(E) = \dfrac{2}{5}$, $P(F) = \dfrac{3}{5}$

Let A be the event 'Student chosen is brilliant'.

Then $P(A / E) = \dfrac{5}{50}$ and $P(A / F) = \dfrac{8}{50}$.

$\therefore$ $P(A) = P(E) . P(A / E) + P(F) . P(A / F) = \dfrac{2}{5} \cdot \dfrac{5}{50} + \dfrac{3}{5} \cdot \dfrac{8}{50} = \dfrac{34}{250}$.

$\therefore$ $P(E / A) = \dfrac{P(E).P(A/E)}{P(E).P(A/E)+P(F).P(A/F)} = \dfrac{5}{17}$.

Solved Example # 36

A pack of cards is counted with face downwards. It is found that one card is missing. One card is drawn and is found to be red. Find the probability that the missing card is red.

Solution

Let A be the event of drawing a red card when one card is drawn out of 51 cards (excluding missing card.) Let A_1 be the event that the missing card is red and A_2 be the event that the missing card is black.

Now by Bayes's theorem, required probability,

$$P(A_1/A) = \frac{P(A_1).(P(A/A_1)}{P(A_1).P(A/A_1)+P(A_2).P(A/A_2)} \quad ....(i)$$

In a pack of 52 cards 26 are red and 26 are black.

Now $P(A_1)$ = probability that the missing card is red $= \dfrac{{}^{26}C_1}{{}^{52}C_1} = \dfrac{26}{52} = \dfrac{1}{2}$

$P(A_2)$ = probability that the missing card is black $= \dfrac{26}{52} = \dfrac{1}{2}$

$P(A/A_1)$ = probability of drawing a red card when the missing card is red.

$= \dfrac{25}{51}$

[$\because$ Total number of cards left is 51 out of which 25 are red and 26 are black as the missing card is red]

Again $P(A/A_2)$ = Probability of drawing a red card when the missing card is black $= \dfrac{26}{51}$

Now from (i), required probability,

$$P(A_1/A) = \frac{\frac{1}{2}\cdot\frac{25}{51}}{\frac{1}{2}\cdot\frac{25}{51}+\frac{1}{2}\cdot\frac{26}{51}} = \frac{25}{51}$$

Solved Example # 37

A bag contains 6 white and an unknown number of black balls (≤ 3). Balls are drawn one by one with replacement from this bag twice and is found to be white on both occassion. Find the probability that the bag had exactly '3' Black balls.

Solution

Apriori, we can think of the following possibilies

(i)	E_1	6W	,	0 B
(ii)	E_2	6W	,	1 B
(iii)	E_3	6W	,	2 B
(iv)	E_4	6W	,	3 B

Clearly $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \dfrac{1}{4}$

Let 'A' be the event that two balls drawn one by one with replacement are both white therefore we have to find

$P\left(\dfrac{E_4}{A}\right)$

By Baye's theorem $P\left(\frac{E_4}{A}\right) = \dfrac{P\left(\frac{A}{E_4}\right)\times P(E_4)}{P\left(\frac{A}{E_1}\right)\times P(E_1)+P\left(\frac{A}{E_2}\right).P(E_2)+P\left(\frac{A}{E_3}\right).P(E_3)+P\left(\frac{A}{E_4}\right).P(E_4)}$

$P\left(\frac{A}{E_4}\right) = \frac{6}{9}\times\frac{6}{9}$; $P\left(\frac{A}{E_3}\right) = \frac{6}{8}\times\frac{6}{8}$; $P\left(\frac{A}{E_2}\right) = \frac{6}{7}\times\frac{6}{7}$;

$P\left(\frac{A}{E_1}\right) = \frac{6}{6}\times\frac{6}{6}$;

Putting values $P\left(\frac{E_4}{A}\right) = \dfrac{\frac{1}{81}}{\frac{1}{81}+\frac{1}{64}+\frac{1}{49}+\frac{1}{36}}$

Self Practice Problems :

23. Box-I contains 3 red and 2 blue balls whilest box-II contains 2 red and 3 blue balls. A fair coin is tossed. If it turns up head, a ball is drawn from box-I, else a ball is drawn from box-II. If the ball drawn is red, then find the probability that the ball is drawn from box-II.

Answer $\frac{3}{5}$

24. Cards of an ordinary deck of playing cards are placed into two heaps. Heap - I consists of all the red cards and heap - II consists of all the black cards. A heap is chosen at random and a card is drawn, if the card drawn is found to be a king, find the probability that the card drawn is from the heap - II.

Answer $\frac{1}{2}$

10. Value of Testimony

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B then P(their combined statement is true) = $\dfrac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}$.

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

However if p is the probability of the happening of the event before their statement, then P(their combined statement is true) = $\dfrac{p\,p_1 p_2}{p\,p_1 p_2 + (1-p)\,(1-p_1)(1-p_2)}$.

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.

If this is not the case and c is the chance of their coincidence testimony then the

Probability that the statement is true = $P\,p_1\,p_2$

Probability that the statement is false = $(1-p).\,c\,(1-p_1)\,(1-p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

Solved Example # 38

A die is thrown, a man C gets a prize of Rs. 5 if the die turns up 1 and gets a prize of Rs. 3 if the die turns up 2, else he gets nothing. A man A whose probability of speaking the truth is $\frac{1}{2}$ tells C that the die has turned up 1 and another man B whose probability of speaking the truth is $\frac{2}{3}$ tells C that the die has turned up 2. Find the expectation value of C.

Solution

Let A and B be the events 'A speaks the truth' and 'B speaks the truth' respectively. Then P(A) = $\frac{1}{2}$ and P(B) = $\frac{2}{3}$.

The experiment consists of three hypothesis

(i) the die turns up 1

(ii) the die turns up 2

(iii) the die turns up 3, 4, 5 or 6

Let these be the events E_1, E_2 and E_3 respectively then $P(E_1) = P(E_2) = \frac{1}{6}$ and $P(E_3) = \frac{4}{6}$.

Let E be the event that the statements made by A and B agree to the same conclusion.

$\therefore \quad P(E / E_1) = P(A) \,.\, P(\bar{B}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$P(E / E_2) = P(\bar{A}) \,.\, P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$

$P(E / E_3) = P(\bar{A}) \,.\, P(\bar{B}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$\therefore \quad P(E) = P(E_1)\, P(E / E_1) + P(E_2)\, P(E / E_2) + P(E_3)\, P(E / E_3)$

$= \frac{1}{6} \,.\, \frac{1}{6} + \frac{1}{6} \,.\, \frac{2}{6} + \frac{4}{6} \,.\, \frac{1}{6} = \frac{7}{36}$

Thus $P(E_1 / E) = \frac{P(E_1)P(E/E_1)}{P(E)} = \frac{1}{7}$

$P(E_2 / E) = \frac{P(E_2)P(E/E_2)}{P(E)} = \frac{2}{7}$

$P(E_3 / E) = \frac{P(E_3)P(E/E_3)}{P(E)} = \frac{4}{7}$

$\therefore \quad$ expectation of C $= \frac{1}{7} \times 5 + \frac{2}{7} \times 3 + 0 = $ Rs. $\frac{11}{7}$

Solved Example #39

A speaks the truth '3 times out of 4' and B speaks the truth '2 times out of 3'. A die is thrown. Both assert that the number turned up is 2. Find the probability of the truth of their assertion.

Solution

Let A and B be the events 'A speaks the truth' and 'B speaks the truth' repectively. Let C be the event 'the number turned up is not 2 but both agree to the same conclustion that the die has turned up 2'.

Then $P(A) = \frac{3}{4}$, $P(B) = \frac{2}{3}$ and $P(C) = \frac{1}{5} \times \frac{1}{5}$

There are two hypotheses

(i) the die turns up 2

(ii) the die does not turns up 2

Let these be the events E_1 and E_2 respectively, then

$P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$ (a priori probabilities)

Now let E be the event 'the statement made by A and B agree to the same conclusion.

then $\quad P(E / E_1) = P(A) \,.\, P(B) = \frac{3}{4} \,.\, \frac{2}{3} = \frac{1}{2}$

$P(E / E_2) = P(\bar{A}) \,.\, P(\bar{B}) \,.\, P(C) = \frac{1}{4} \,.\, \frac{1}{3} \,.\, \frac{1}{25} = \frac{1}{300}$

Thus $P(E) = P(E_1)\, P(E / E_1) + P(E_2)\, P(E / E_2)$

$= \frac{1}{6} \times \frac{1}{2} + \frac{5}{6} \times \frac{1}{300} = \frac{31}{360}$

$\therefore \quad P(E_1 / E) = \frac{P(E_1)\,P(E/E_1)}{P(E)} = \frac{30}{31}$

Self Practice Problems :

25. A ball is drawn from an urn containing 5 balls of different colours including white. Two men A and B whose probability of speaking the truth are $\frac{1}{3}$ and $\frac{2}{5}$ respectively assert that the ball drawn is white. Find the probability of the truth of their assertion.

Answer $\frac{4}{7}$

11. Binomial Probability Distribution :

(i) A probability distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(ii) Mean of any probability distribution of a random variable is given by :

$$\mu = \frac{\Sigma\, p_i\, x_i}{\Sigma\, p_i} = \Sigma\, p_i\, x_i \quad (\text{Since } \Sigma\, p_i = 1)$$

(iii) Variance of a random variable is given by, $\sigma^2 = \Sigma\, (x_i - \mu)^2 \,.\, p_i$

$\therefore \quad \sigma^2 = \Sigma\, p_i\, x_i^2 - \mu^2$ (Note that SD = $+\sqrt{\sigma^2}$)

(iv) The probability distribution for a binomial variate 'X' is given by :
$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$ where P(X = r) is the probability of r successes.

The recurrence formula $\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1} \cdot \frac{p}{q}$, is very helpful for quickly computing P(1) . P(2) . P(3) etc. if P(0) is known.

(v) Mean of BPD = np ; variance of BPD = npq.

(vi) If p represents a person's chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

Solved Example # 40

A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine

(i) k (ii) $P(X < 3)$ (iii) $P(X > 6)$ (iv) $P(0 < X < 3)$

[**Hint :** Use $\sum P(X) = 1$ to determine k, $P(X < 3) = P(0) + P(1) + P(2)$, $P(X > 6) = P(7)$ etc.]

Solved Example # 41

A pair of dice is thrown 5 times. If getting a doublet is considered as a success, then find the mean and variance of successes.

Solution

In a single throw of a pair of dice, probability of getting a doublet = $\frac{1}{6}$

con sidering it to be a success, p = $\frac{1}{6}$

$\therefore \quad q = 1 - \frac{1}{6} = \frac{5}{6}$

mean = $5 \times \frac{1}{6} = \frac{5}{6}$

variance = $5 \times \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{36}$

Solved Example # 42

A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of successes.

Solution

p = probability of getting a total of 9 = $\frac{4}{36} = \frac{1}{9}$

$\therefore \quad q = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore \quad$ mean = np = $4 \times \frac{1}{9} = \frac{4}{9}$

variance = npq = $4 \times \frac{1}{9} \times \frac{8}{9} = \frac{32}{81}$

Solved Example # 43

Difference between mean and variance of a Binomial variate is '1' and difference between their squares is '11'. Find the probability of getting exactly three success

Solution

Mean = np & variance = npq

therefore, $np - npq = 1$ ....(i)

$n^2p^2 - n^2p^2q^2 = 11$ ....(ii)

Also, we know that p + q = 1 ....(iii)

Divide equation (ii) by square of (i) and solve, we get, $q = \frac{5}{6}$, $p = \frac{1}{6}$ & n = 36

Hence probability of '3' success = ${}^{36}C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{33}$ Ans.

Self Practice Problems :

26. A box contains 2 red and 3 blue balls. Two balls are drawn successively without replacement. If getting 'a red ball on first draw and a blue ball on second draw' is considered a success, then find the mean and variance of successes.

Answer mean = 2.1, σ^2 = .63

27. Probability that a bulb produced by a factory will fuse after an year of use is 0.2. If fusing of a bulb is considered an failure, find the mean and variance of successes for a sample of 10 bulbs.
Answer mean = 8 and variance = 1.6

28. A random variable X is specified by the following distribution law :

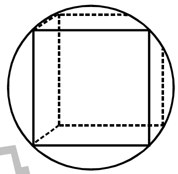
X	2	3	4
P(X = x)	0.3	0.4	0.3

Then the variance of this distribution is :
(A*) 0.6 (B) 0.7 (C) 0.77 (D) 1.55

12. Geometrical Applications:
The following statements are axiomatic :

(i) If a point is taken at random on a given straight line segment AB, the chance that it falls on a particular segment PQ of the line segment is PQ/AB.
(ii) If a point is taken at random on the area S which includes an area σ, the chance that the point falls on σ is σ/S.

Solved Example # 44
A sphere is circumscribed over a cube. Find the probability that a point lies inside the sphere, lies outside the cube.
Solution



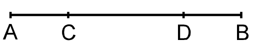
Required probability = $\dfrac{\text{favorable volume}}{\text{total volume}}$

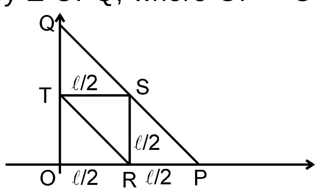
Clearly if edge length of cube is a radius of sphere will be $\dfrac{a\sqrt{3}}{2}$

Thus, volume of sphere = $\dfrac{4}{3}\pi\left(\dfrac{a\sqrt{3}}{2}\right)^3 = \dfrac{\pi a^3\sqrt{3}}{2}$

Hence P = $1 - \dfrac{1}{\pi\dfrac{\sqrt{3}}{2}} = 1 - \dfrac{2}{\pi\sqrt{3}}$

Solved Example # 45
A given line segment is divided at random into three parts. What is the probability that they form sides of a possible triangle ?
Solution
Let AB be the line segment of length ℓ.
Let C and D be the points which divide AB into three parts.
Let AC = x, CD = y. Then DB = ℓ – x – y.
Clearly x + y < ℓ
$\therefore$ the sample space is given by
the region enclosed by Δ OPQ, where OP = OQ = ℓ





Area of ΔOPQ = $\dfrac{\ell^2}{2}$
Now if the parts AC, CD and DB form a triangle, then

$x + y > \ell - x - y$ i.e. $x + y > \dfrac{\ell}{2}$ ....(i)

$x + \ell - x - y > y$ i.e. $y < \dfrac{\ell}{2}$ ....(ii)

$y + \ell - x - y > x$ i.e. $x < \dfrac{\ell}{2}$ ....(iii)

from (i), (ii) and (iii), we get
the event is given by the region closed in ΔRST

$\therefore$ Probability of the event = $\dfrac{ar(\Delta RST)}{ar(\Delta OPQ)} = \dfrac{\frac{1}{2}\cdot\frac{\ell}{2}\cdot\frac{\ell}{2}}{\frac{\ell^2}{2}} = \dfrac{1}{4}$

Solved Example # 46

On a line segment of length L two points are taken at random, find the probability that the distance between them is ℓ , where $\ell < 1$

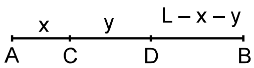
Solution

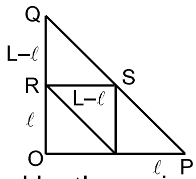
Let AB be the line segment

Let C and D be any two points on AB so that AC = x and CD = y. Then $x + y < L$, $y > \ell$

$\therefore$ sample space is represented by the region enclosed by ΔOPQ.

Area of $\Delta OPQ = \frac{1}{2}L^2$





The event is represented by the region, bounded by the ΔRSQ

Area of $\Delta RSQ = \frac{1}{2}(L - \ell)^2$

$\therefore$ probability of the event = $\left(\dfrac{L-\ell}{L}\right)^2$

Self Practice Problems :

29. A line segment of length a is divided in two parts at random by taking a point on it, find the probability that no part is greater than b, where 2b > a

Answer $\dfrac{2b-a}{a}$

30. A cloth of length 10 meters is to be randomly distributed among three brothers, find the probability that no one gets more than 4 meters of cloth.

Answer $\dfrac{1}{25}$